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Determination of the Exact Optimum Order Statistics for Estimating the Parameters of the Exponential Distribution from Censored Samples*

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This paper presents the small sample optimum choice of the $k(\leq r_2)$ order statistics for the best linear unbiased estimate (BLUE) of the parameters μ and σ or σ alone (μ known) when the sample is Type II censored on the right. For $n = 2(1)10$, $k = 1(1)r_2$ and $r_2 = \{[.50n]+1\}\ (1)n$, the optimum ranks, the coefficients of the BLUEs have been presented in Table I.

INTRODUCTION

Assume that we are sampling from the exponential distribution

$$f(x) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma} \qquad x \geq \mu; \quad \sigma > 0 \tag{1.1}$$

under Type II censoring procedure with a fixed proportion of censoring on the right. The symbols μ and σ denote the location and the scale parameters respectively. From a random sample of size n let $x_{(1)} < x_{(2)} < \cdots < x_{(r_2)}$ be the uncensored portion of the sample. The integer r_2 is the rank of the largest observation in the uncensored portion of the sample. We are interested in the estimation of μ and σ or σ alone, when μ is known, on the basis of k suitably chosen order statistics $x_{(n_1)}, x_{(n_2)}, \cdots x_{(n_k)}$ where $n_1, n_2, \cdots n_k$ are the ranks satisfying the inequality $1 \leq n_1 < \cdots < n_k \leq r_2$.

The small sample as well as the asymptotic ($n \to \infty$) situation in uncensored samples (i.e., all the n observations are available for estimation) has been considered by Harter (1961), Kulldorff (1963a, 1963b), Ogawa (1960), Saleh (1964), Saleh and Ali (1966a), Sarhan, Greenberg and Ogawa (1963) and Siddiqui (1963). The asymptotic ($n \to \infty$) situation concerning the optimum choice of the $k(\leq r_2)$ order statistics has been considered by Saleh (1964, 1966b).

We shall present here the small sample optimum choice of the $k(\leq r_2)$ order statistics for the best linear unbiased estimate (BLUE) of the parameters μ and σ or σ alone (μ known) when the sample is Type II censored on the right. For $n = 2(1)10$, $k = 1(1)r_2$ and $r_2 = \{[.50n] + 1\}(1)n$, the optimum ranks, the coefficients of the BLUEs have been presented in Table I.

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Table I: *The Coefficients of Blue of the Scale Parameter Based on Optimum Order Statistics From Censored Samples*

$n = 2$

r_2	k	n_1	n_2	K^*
1	1	1		1.0000
		2.0000		
2	1	2		1.8000
		.6667		
2	2	1	2	2.0000
		.5000	.5000	

$n = 3$

r_2	k	n_1	n_2	n_3	K^*
2	1	2			1.9231
		.5455			
	2	2	3		2.9231
		.4474	.3421		
3	1	3			2.4694
		.5455			
	2	2	3		2.9231
		.4474	.3421		
	3	1	2	3	3.0000
		.3333	.3333	.3333	

TABLE I *Continued*

$n = 4$

r_2	k	n_1	n_2	n_3	n_4	K^*
2	1	2				1.9600
		1.7143				
	2	1	2			2.0000
		.5000	1.5000			
3	1	3				2.7705
		.9231				
	2	2	3			2.9600
		.4595	.6757			
	3	1	2	3		3.0000
		.3333	.3333	.6667		
4	1	4				3.0488
		.4800				
	2	3	4			3.7705
		.4130	.2652			
	3	2	3	4		3.9600
		.3434	.2525	.2525		
	4	1	2	3	4	4.0000
		.2500	.2500	.2500	.2500	

* The original calculations were carried with precision to seven decimal places.

$n = 5$

r_2	k	n_1	n_2	n_3	n_4	n_5	K^*
3	1	3					2.8726
		1.2766					
	1	2	3				2.9756
		.4672	1.0082				
	3	1	2	3			3.0000
		.3333	.3333	1.0000			
4	1	4					3.5524
		.7792					
	2	2	4				3.8987
		.5342	.5919				
	3	2	3	4			3.9756
		.3497	.2515	.5031			
5	1	5					3.5622
		.4380					
	2	3	5				4.6726
		.5280	.2568				
	3	2	4	5			4.8987
		.4251	.2669	.2041			
	4	2	3	4	5		4.9757
		.2794	.2010	.2010	.2010		
	5	1	2	3	4	5	5.0000
		.2000	.2000	.2000	.2000	.2000	

TABLE I *Continued*

$n = 6$

r_2	k	n_1	n_2	n_3	n_4	n_5	n_6	K^*
3	1	3						2.4190
		1.6216						
	2	2	3					2.9836
		.4725	1.3407					
	3	1	2	3				3.0000
		.3333	.3333	1.3333				
4	1	4						4.2787
		1.0526						
	2	2	4					3.9436
		.5198	.8520					
	3	2	3	4				3.9836
		.3539	.2510	.7531				
	4	1	2	3	4			4.9836
		.2500	.2500	.2500	.7500			
5	1	5						4.2787
		.6897						
	2	3	5					5.5388
		.5010	.4766					
	3	2	4	5				4.9436
		.4146	.2751	.4046				
	4	2	3	4	5			4.9836
		.2829	.2007	.2007	.4013			
	5	1	2	3	4	5		5.0000
		.2000	.2000	.2000	.2000	.4000		
6	1	5						4.2787
		.6897						
	2	4	6					5.5388
		.4939	.2167					
	3	3	5	6				5.8421
		.4152	.2238	.1712				
	4	2	4	5	6			5.9436
		.3449	.2288	.1682	.1682			
	5	2	3	4	5	6		5.9836
		.2356	.1671	.1671	.1671	.1671		
	6	1	2	3	4	5	6	6.0000
		.1667	.1667	.1667	.1667	.1667	.1667	

TABLE I *Continued*

$n = 7$

r_2	k	n_1	n_2	n_3	n_4	n_5	n_6	n_7	K^*
4	1	4 1.3166							3.8283
	2	2 .5130	4 1.1076						3.9638
	3	2 .3569	3 .2507	4 1.003					3.9882
	4	1 .2500	2 .2500	3 .2500	4 1.0000				4.0000
5	1	5 .9150							4.5621
	2	3 .4930	5 .6852						4.9040
	3	2 .4096	4 .2801	5 .6044					4.9638
	4	2 .2854	3 .2005	4 .2005	5 .6014				4.9882
	5	1 .2000	2 .2000	3 .2000	4 .2000	5 .6000			5.0000
6	1	6 .6278							4.9574
	2	4 .4751	6 .4012						5.7514
	3	3 .4095	5 .2304	6 .3388					5.9039
	4	2 .3409	4 .2331	5 .1677	6 .3354				5.9638
	5	2 .2377	3 .1670	4 .1670	5 .1670	6 .3340			5.9882
	6	1 .1667	2 .1667	3 .1667	4 .1667	5 .1667	6 .3333		6.0000
7	1	6 .6278							4.9574
	2	5 .4675	7 .1886						6.3621
	3	4 .4048	6 .1937	7 .1481					6.7514
	4	3 .3502	5 .1970	6 .1448	7 .1448				6.9039
	5	2 .2920	4 .1996	5 .1436	6 .1436	7 .1436			6.9638
	6	2 .2037	3 .1431	4 .1431	5 .1431	6 .1431	7 .1431		6.9882
	7	1 .1429	2 .1429	3 .1429	4 .1429	5 .1429	6 .1429	7 .1429	7.0000

TABLE I *Continued*

$n = 8$

r_2	k	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	K^*
4	1	4								3.8784
		1.5760								
	2	2	4							3.9748
		.5092	1.3610							
	3	2	3	4						3.9912
		.3592	.2506	1.2528						
	4	1	2	3	4					4.0000
		.2500	.2500	.2500	1.2500					
5	1	5								4.7043
		1.1305								
	2	3	5							4.9345
		.4903	.8897							
	3	2	4	5						4.9748
		.4068	.2834	.8041						
	4	2	3	4	5					4.9912
		.2872	.2004	.2004	.8014					
	5	1	2	3	4	5				5.0000
		.2000	.2000	.2000	.2000	.8000				
6	1	6								5.3463
		.8211								
	2	4	6							5.8384
		.4714	.5755							
	3	2	4	6						5.9348
		.3410	.3454	.5662						
	4	2	4	5	6					5.9748
		.3387	.2360	.1674	.5021					
	5	2	3	4	5	6				5.9912
		.2393	.1669	.1669	.1669	.5007				
	6	1	2	3	4	5	6			6.0000
		.1667	.1667	.1667	.1667	.1667	.5000			
7	1	7								5.5952
		.5821								
	2	4	7							6.6489
		.5347	.3846							
	3	3	5	7						6.8489
		.3528	.3037	.3365						
	4	2	4	6	7					6.9348
		.2918	.2956	.1961	.2884					
	5	2	4	5	6	7				6.9748
		.2902	.2021	.1434	.1434	.2867				
	6	2	3	4	5	6	7			6.9912
		.2051	.1430	.1430	.1430	.1430	.2861			
	7	1	2	3	4	5	6	7		7.0000
		.1429	.1429	.1429	.1429	.1429	.1429	.2857		
8	1	7								5.5952
		.5821								
	2	5	8							7.1737
		.5536	.1878							
	3	4	7	8						7.6489
		.4648	.2036	.1307						
	4	3	5	7	8					7.8576
		.3079	.2650	.1664	.1273					

TABLE I *Continued*

$n = 8$ Continued —

r_2	k	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	K^*
8	5	2	4	6	7	8				7.9348
		.2551	.2583	.1714	.1260	.1260				
	6	2	4	5	6	7	8			7.9748
		.2538	.1768	.1254	.1254	.1254	.1254			
	7	2	3	4	5	6	7	8		7.9912
		.1794	.1251	.1251	.1251	.1251	.1251	.1251		
	8	1	2	3	4	5	6	7	8	8.0000
		.1250	.1250	.1250	.1250	.1250	.1250	.1250	.1250	

$n = 9$

r_2	k	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	K^*
5	1	5									4.7864
		1.3411									
	2	3	5								4.9522
		.4894	1.0924								
	3	2	4	5							4.9813
		.4051	.2858	1.0037							
	4	2	3	4	5						4.9931
		.2887	.2003	.2003	1.0014						
	5	1	2	3	4	5					5.0000
		.2000	.2000	.2000	.2000	1.0000					
6	1	6									5.5486
		1.0044									
	2	3	6								5.8876
		.5265	.8040								
	3	2	4	6							5.9569
		.3387	.3413	.7370							
	4	2	4	5	6						5.9313
		.3374	.2380	.1672	.6637						
	5	2	3	4	5	6					5.9931
		.2405	.1669	.1669	.1669	.6674					
	6	1	2	3	4	5	6				6.0000
		.1667	.1667	.1667	.1667	.1667	.6667				
7	1	7									6.0951
		.7525									
	2	4	7								6.7818
		.5157	.5407								
	3	3	5	7							6.9122
		.3506	.2966	.4861							
	4	2	4	6	7						6.9569
		.2900	.2923	.1998	.4312						
	5	2	4	5	6	7					6.9813
		.2890	.2039	.1432	.1432	.4297					
	6	2	3	4	5	6	7				6.9931
		.2061	.1430	.1430	.1430	.1430	.4290				
	7	1	2	3	4	5	6	7			7.0000
		.1429	.1429	.1429	.1429	.1429	.1429	.4286			

TABLE I *Continued*

n = 9 Continued—

r_2	k	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	K^*
8	1	8									6.1973
		.5468									
	2	5	8								7.5569
		.5110	.3384								
	3	3	6	8							7.8106
		.3969	.3106	.2955							
	4	3	5	7	8						7.9122
		.3063	.2591	.1719	.2528						
	5	2	4	6	7	8					7.9569
		.2536	.2555	.1747	.1257	.2514					
	6	2	4	5	6	7	8				7.9813
		.2528	.1784	.1253	.1253	.1253	.2506				
	7	2	3	4	5	6	7	8			7.9931
		.1803	.1251	.1251	.1251	.1251	.1251	.2502			
	8	1	2	3	4	5	6	7	8		8.0000
		.1250	.1250	.1250	.1250	.1250	.1250	.1250	.2500		
9	1	8									6.1973
		.5468									
	2	6	9								8.0180
		.5271	.1680								
	3	4	7	9							8.5818
		.4076	.2875	.1398							
	4	3	6	8	9						8.8106
		.3518	.2753	.1484	.1135						
	5	3	5	7	8	9					8.9122
		.2719	.2300	.1526	.1122	.1122					
	6	2	4	6	7	8	9				8.9569
		.2253	.2270	.1552	.1116	.1116	.1116				
	7	2	4	5	6	7	8	9			8.9813
		.2247	.1585	.1113	.1113	.1113	.1113	.1113			
	8	2	3	4	5	6	7	8	9		8.9931
		.1603	.1112	.1112	.1112	.1112	.1112	.1112	.1112		
	9	1	2	3	4	5	6	7	8	9	9.0000
		.1111	.1111	.1111	.1111	.1111	.1111	.1111	.1111	.1111	

TABLE I *Continued*

$n = 10$

r_2	k	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	K^*
5	1	5										4.8382
		1.5489										
	2	3	5									4.9634
		.4892	1.2942									
	3	2	4	5								4.9856
		.4039	.2876	1.2035								
	4	2	3	4	5							4.9945
		.2898	.2002	.2002	1.2013							
	5	1	2	3	4	5						5.0000
		.2000	.2000	.2000	.2000	1.2000						
6	1	6										5.6683
		1.1825										
	2	3	6									5.9192
		.5193	.9761									
	3	2	4	6								5.9692
		.3374	.3390	.9063								
	4	2	4	5	6							5.9856
		.3365	.2395	.1671	.8353							
	5	2	3	4	5	6						5.9945
		.2415	.1668	.1668	.1668	.8341						
	6	1	2	3	4	5	6					6.0000
		.1667	.1667	.1667	.1667	.1667	.8333					
7	1	7										6.3630
		.9127										
	2	4	7									6.8487
		.5068	.6912									
	3	3	5	7								6.9391
		.3500	.2930	.6327								
	4	2	4	6	7							6.9692
		.2890	.2904	.2023	.5740							
	5	2	4	5	6	7						6.9856
		.2883	.2052	.1432	.1432	.5726						
	6	2	3	4	5	6	7					6.9945
		.2070	.1430	.1430	.1430	.1430	.5719					
	7	1	2	3	4	5	6	7				7.0000
		.1429	.1429	.1429	.1429	.1429	.1429	.5714				
8	1	8										6.8118
		.6998										
	2	5	8									7.7108
		.4963	.4756									
	3	3	6	8								7.8792
		.3901	.3069	.4264								
	4	3	5	7	8							7.9391
		.3059	.2561	.1751	.3779							
	5	2	4	6	7	8						7.9692
		.2527	.2540	.1769	.1255	.3764						
	6	2	4	5	6	7	8					7.9856
		.2522	.1795	.1252	.1252	.1252	.3757					
	7	2	3	4	5	6	7	8				7.9945
		.1811	.1251	.1251	.1251	.1251	.1251	.3753				
	8	1	2	3	4	5	6	7	8			8.0000
		.1250	.1250	.1250	.1250	.1250	.1250	.1250	.3750			

TABLE I *Continued*

n = 10 Continued

r_2	k	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	K^*
9	1	8										6.8118
		.6998										
	2	6	9									8.4388
		.4913	.3030									
	3	4	7	9								8.7717
		.3957	.2765	.2631								
	4	3	6	8	9							8.8792
		.3462	.2723	.1532	.2252							
	5	3	5	7	8	9						8.9391
		.2717	.2275	.1555	.1119	.2237						
	6	2	5	6	7	8	9					8.9692
		.2245	.2256	.1572	.1115	.1115	.2230					
	7	2	4	5	6	7	8	9				8.9856
		.2241	.1595	.1113	.1113	.1113	.1113	.2226				
	8	2	3	4	5	6	7	8	9			8.9945
		.1609	.1112	.1112	.1112	.1112	.1112	.1112	.2224			
	9	1	2	3	4	5	6	7	8	9		9.0000
		.1111	.1111	.1111	.1111	.1111	.1111	.1111	.1111	.2222		
10	1	8										6.8118
		.6998										
	2	7	10									8.8324
		.5050	.1525									
	3	5	8	10								9.5108
		.4023	.2594	.1262								
	4	4	7	9	10							9.7717
		.3552	.2482	.1338	.1023							
	5	3	6	8	9	10						9.8792
		.3112	.2447	.1377	.1012	.1012						
	6	3	5	7	8	9	10					9.9391
		.2443	.2046	.1399	.1006	.1006	.1006					
	7	2	4	6	7	8	9	10				9.9692
		.2020	.2030	.1414	.1003	.1003	.1003	.1003				
	8	2	4	5	6	7	8	9	10			9.9856
		.2017	.1435	.1001	.1001	.1001	.1001	.1001	.1001			
	9	2	3	4	5	6	7	8	9	10		9.9945
		.1448	.1001	.1001	.1001	.1001	.1001	.1001	.1001	.1001		
	10	1	2	3	4	5	6	7	8	9	10	10.0000
		.1000	.1000	.1000	.1000	.1000	.1000	.1000	.1000	.1000	.1000	

OPTIMUM RANKS OF ORDER STATISTICS FOR THE ESTIMATION OF σ (μ KNOWN)

For $k(\leq r_2)$ let $x_{(n_1)}, x_{(n_2)}, \cdots x_{(n_k)}$ be the k order statistics of fixed ranks $n_1, n_2, \cdots, n_k$ satisfying the inequality $1 \leq n_1 < \cdots < n_k \leq r_2$. Then the BLUE of σ due to Kuldorff (1963) is, $\hat{\sigma} = b_0\mu + \sum_{i=1}^{k} b_i x_{(n_i)}$ where

$$b_i = (\delta_{1,i}/\delta_{2,i} - \delta_{1,i+1}/\delta_{2,i+1})/K \quad \text{for} \quad i = 0, 1, \cdots k$$

and

$$\delta_{ri} = \sum_{j=n_{i-1}}^{n_i - 1} (n - j)^{-r} \qquad (r = 1, 2; i = 1, 2, \cdots k);$$

$$K = \sum_{i=1}^{k} \delta_{1,i}^2/\delta_{2,i} .$$

The variance of $\hat{\sigma}$ is $V(\hat{\sigma}) = \sigma^2/K$.

In order to determine the optimum ranks of the order statistics, we minimize the variance of σ or, equivalently, maximize K subject to the restriction $1 \leq n_1 < \cdots < n_k \leq r_2$. The maximization of K subject to the restriction $1 \leq n_1 < \cdots < n_k \leq n$ has been considered by Kulldorff (1963b). Let this maximum be attained at $(n_0^1 < n_2^0 < \cdots < n_k^0)$. For the maximum of K we have two situations (i) the proportion of censoring is such that $r_2 \geq n_k^0$; (ii) the proportion of censoring is such that $r_2 < n_k^0$. In the first case, the optimum ranks are $n_1^0, \cdots, n_k^0$. In the second case, we have solved the problem numerically on an IBM 7040 for $n = 2(1)10$, $r_2 = \{[.50n] + 1\}(1)n$ and $k = 1(1)r_2$. The optimum ranks, the coefficients of the BLUE and the maximum value of K are presented in Table I. If we use the optimum ranks, the relative efficiency (RE) of this BLUE of σ compared to the BLUE based on all the observations in the censored sample (Sarhan and Greenberg, 1957) is K^*/r_2, where K^* is the maximum of K, subject to the restriction $1 \leq n_1 < \cdots < n_k \leq r_2$.

Optimum Ranks of Order Statistics for the Estimation of μ and σ

For $k(\leq r_2)$ let $x_{(n_1)}, \cdots, x_{(n_k)}$ be k order statistics of fixed ranks $n_1, \cdots, n_k$ satisfying the inequality $1 \leq n_1 < \cdots < n_k \leq r_2$. Then the BLUE's of σ and μ due to Kulldorff (1963b) are:

$$\hat{\sigma} = \sum_{i=1}^{k} b_i x_{(n_i)}$$

$$\hat{\mu} = x_{(n_1)} - \hat{\sigma}\, \delta_{11}$$

where

$$b_i = -\delta_{12}/L\, \delta_{22} \quad \text{for} \quad i = 1$$

$$= (\delta_{1,i}/\delta_{2,i} - \delta_{1,i+1}/\delta_{2,i+1})/L \quad \text{for} \quad i = 2, \cdots k$$

and

$$L = \sum_{i=2}^{k} \delta_{1,i}^2/\delta_{2,i}\,.$$

The variances and covariance of the BLUE's are

$$V(\hat{\sigma}) = \sigma^2/L$$

$$V(\hat{\mu}) = \sigma^2(\delta_{21} + \delta_{11}^2/L)$$

$$\text{Cov}\,(\hat{\mu}, \hat{\sigma}) = -\sigma^2\, \delta_{11}/L.$$

The generalized variance of the estimates $\hat{\mu}$ and $\hat{\sigma}$ is

$$\Lambda(\hat{\mu}, \hat{\sigma}) = V(\hat{\mu})V(\hat{\sigma}) - \text{Cov}^2\,(\hat{\mu}, \hat{\sigma})$$

$$= \sigma^4\, \delta_{21}/L.$$

In order to determine the optimum ranks of the order statistics, we minimize the generalized variance $\sigma^4\delta_{21}/L$ with respect to $n_1, n_2, \cdots n_k$ or equivalently maximize $L(\delta_{21})^{-1}$ subject to the restriction $1 \leq n_1 < \cdots < n_k \leq r_2$. By theorem 7 (Kulldorff, 1963b), it is clear that $L(\delta_{21})^{-1}$ as a function n_1 decreases;

therefore we choose $n_1 = 1$. Let us now turn to the second step of maximizing $L(\delta_{21})^{-1}$ with respect to $n_2, \cdots, n_k$ while keeping $n_1 = 1$ fixed. If we substitute $n - 1$ for n, $k - 1$ for k and $n_{i+1} - 1$ for n_i' $(i = 1, 2, \cdots, k - 1)$ in the expression for K in Section II we obtain

$$K' = \sum_{i=1}^{k-1} \frac{\delta_{1i}'^2}{\delta_{2i}'}$$

where

$$\delta_{ri}' = \sum_{j=n'_{i-1}}^{n'_{i+1}-1} (n - 1 - j)^{-r} \quad \cdot \quad (r = 1, 2; i = 1, 2, \cdots k - 1)$$

Therefore $L = n^2K'$, where K' is a function of $n_1', n_2', \cdots n_{k-1}'$ satisfying the inequality $1 \leq n_1' < \cdots < n_{k-1}' \leq (r_2 - 1)$. The remaining problem of maximizing K' with respect to $n_1' \cdots n_{k-1}'$ subject to $1 \leq n_1' < \cdots < n_{k-1}' \leq (r_2 - 1)$ has been discussed in Section II and we encounter two situations (i) $(r_2 - 1) \geq n_{k-1}^*$ (ii) $(r_2 - 1) < n_{k-1}^*$ where n_{k-1}^* is the largest rank in selecting $k - 1$ order statistics from an uncensored sample of size $n - 1$.

In the first case, the optimum ranks coincide with the ranks when the sample is not censored. In the second case, we maximize K' with respect to $n_1' \cdots n_{k-1}'$ numerically subject to the restriction $1 \leq n_1' < \cdots < n_{k-1}' \leq (r_2 - 1)$. Let these ranks be $n_1^{**}, \cdots, n_{k-1}^{**}$. Then the optimum ranks of the k order statistics for the estimation of μ and σ simultaneously are

$$1, (n_1^{**} + 1), (n_2^{**} + 1), \cdots (n_{k-1}^{**} + 1)$$

and the estimates are

$$\hat{\mu} = x_{(1)} - \hat{\sigma}/n$$

$$\hat{\sigma} = -\left(\sum_{i=1}^{k-1} b_i^{**}\right)x_{(1)} + \sum_{i=1}^{k-1} b_i^{**}x_{(n_i^{**}+1)}$$

where the coefficients b_i^{**} $(i = 1, \cdots k - 1)$ are based on the Ranks 1 and $(n_i^{**} + 1)$ $(i = 1, 2 \cdots k - 1)$. If we use these ranks, the joint efficiency (JE) of the estimates and the relative efficiencies (RE) of the estimates of μ and σ are

$$\text{JE}\ (\hat{\mu}, \hat{\sigma}) = \frac{K^*}{(r_2 - 1)}$$

$$\text{RE}\ (\hat{\sigma}) = \frac{K^*}{(r_2 - 1)}$$

$$\text{RE}\ (\hat{\mu}) = \frac{r_2K^*}{(r_2 - 1)(K^* + 1)}$$

where K^* is the maximum of K'.

We present two examples for the estimation of σ when μ is known and that of μ and σ when both are unknown.

Example 1: (Estimation of σ when μ is known). Assume $n = 9$, $r_2 = 7$ and $k = 4$. From Table I, we obtain the optimum ranks as 2, 4, 6 and 7. The BLUE of σ based on the corresponding order statistics is given by

$$\hat{\sigma} = -1.2133u + .2900x_{(2)} + .2923x_{(4)} + .1998x_{(6)} + .4312x_{(7)}\ .$$

The coefficient of μ is the negative of the sum of the coefficients of the order statistics in the above estimate. The Relative efficiency of the estimate is given by

$$\text{RE}(\hat{\sigma}) = \frac{6.9569}{7.0000} = 99.38\%$$

Example 2: (Estimation of μ and σ). Assume $n = 9$, $r_2 = 7$ and $k = 5$. From the reasonings of Section III we first select the first order statistics $x_{(1)}$. Then use Table I for $n = 8$, $r_2 = 6$ and $K = 4$ to obtain the optimum ranks as 2, 4, 5 and 6. Thus, the ranks of the relevent optimum order statistics to be used for estimation of u and σ are 1, 3, 5, 6 and 7. The BLUEs are given by

$$\hat{\mu} = x_{(1)} - \hat{\sigma}/n$$

$$\hat{\sigma} = -1.2442x_{(1)} + .3387x_{(3)} + .2360x_{(5)} + .1674x_{(6)} + .5021x_{(7)}$$

Note that the coefficients of the estimate of σ are available in Table I for $n = 8$, $r_2 = 6$ and $K = 4$. The coefficient of $x_{(1)}$ is the negative sum of the coefficients of the remaining order statistics used in the estimate of σ.

The joint efficiency and relative efficiencies of the estimates are

$$\text{JE}(\hat{\mu}, \hat{\sigma}) = \frac{5.9746}{6} = 99.42\%$$

$$\text{RE}(\hat{\sigma}) = \frac{5.9746}{6} = 99.42\%$$

$$\text{RE}(\hat{\mu}) = \frac{7 \times 5.9746}{6 \times 6.9746} = 99.70\%.$$

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NOTE: While reviewing this paper the author has extensively computed the Tables of optimum ranks, coefficients of linear estimates and the maximum value of k for all possible right censoring. The results have been deposited with the Mathematics of Computation as unpublished Mathematical tables under the title "Tables for estimation of exponential distribution by linear combination of the optimal subset of order statistics by F. Zabransky, M. Sibuya and A. K. Md. Ehsanes Saleh."

References

1. Harter, H. L., 1961. Estimating the parameters of negative exponential populations from one or two order statistics, *Ann. Math. Statist.*, *32*, pp. 1078–1090.
2. Kulldorff, G., 1963a. On the optimum spacings for the exponential distribution, Final mimeographed report, University of Lund, Sweden.
3. Kulldorff, G., 1963b. Estimation of one or two parameters of the exponential distribution on the basis of suitably chosen order statistics, *Ann. Math. Statist.*, *34*, pp. 1419–1431.

4. Ogawa, J., 1960. Determination of optimum spacings for the estimation of the scale parameters of an exponential distribution based on sample quantiles, *Ann. Inst. Statist. Math. Tokyo*, *12*, pp. 155–141.
5. Saleh, A. K. Md. Ehsanes, 1964. On the estimation of the parameters of the exponential distribution based on optimum order statistics in censored samples, Ph.D. Dissertation, University of Western Ontatio, London, Canada.
6. Saleh, A. K. Md. Ehsanes and Mir M. Ali, 1966a. Asymptotic optimum quantiles for the estimation of the parameters of the negative exponential distribution, *Ann. Math. Statist.* *37*, pp. 143–151.
7. Saleh, A. K. Md. Ehsanes, 1966b. Estimation of the parameters of the exponential distribution based on optimum order statistics in censored samples, *Ann. Math. Statist.*, *37*, pp. 1717–1735.
8. Sarhan, A. E. and Greenberg, B. G., 1957. Tables for best linear estimates by order statistics of the parameters of single exponential distributions from singly and doubly censored samples, *J. Amer. Statist. Assn.*, *52*, pp. 58–87.
9. Sarhan, A. E., Greenberg, B. G. and Ogawa, J., 1963. Simplified estimates for the exponential distribution, *Ann. Math. Statist.*, *34*, pp. 102–116.
10. Siddiqui, M. M., 1963. Optimum estimators of the parameters of negative exponential distribution from one or two order statistics. *Ann. Math. Statist.*, *34*, pp. 117–121.